On similarity solutions for boundary layer flows with prescribed heat flux

BERNARD BRIGHI† and JEAN-DAVID HOERNEL‡

Université de Haute-Alsace, Laboratoire de Mathématiques et Applications

4 rue des frères Lumière, 68093 MULHOUSE (France)

Abstract

This paper is concerned with existence, uniqueness and behavior of the solutions of the autonomous third order nonlinear differential equation $f''' + (m + 2) f f'' - (2m + 1) f'^2 = 0$ on $\mathbb{R}^+$ with the boundary conditions $f(0) = -\gamma$, $f'(\infty) = 0$ and $f''(0) = -1$. This problem arises when looking for similarity solutions for boundary layer flows with prescribed heat flux. To study solutions we use some direct approach as well as blowing-up coordinates to obtain a plane dynamical system.

1 Introduction

We consider the following third order non-linear autonomous differential equation found in [8]

$$f''' + (m + 2) f f'' - (2m + 1) f'^2 = 0 \tag{1.1}$$

with the boundary conditions

$$f(0) = -\gamma, \tag{1.2}$$

$$f'(\infty) = 0, \tag{1.3}$$

$$f''(0) = -1 \tag{1.4}$$

where $f'(\infty) := \lim_{t \to \infty} f'(t)$.

This equation gives the similarity solutions for free convection boundary-layer flows along a vertical permeable surface with prescribed surface heating and mass transfer rate. The solutions depend on two parameters: m, the power-law exponent and γ, the mass transfer parameter. The case $\gamma = 0$ corresponds to an impermeable wall, $\gamma < 0$ to a fluid suction and $\gamma > 0$ to a fluid injection. In the following we are investigating for existence and uniqueness of the solutions of (1.1)-(1.4) according to the values of m and γ. We also gives some results about the boundedness and behavior of the solutions.

The problem involving similarity solutions with prescribed surface temperature leads to a similar equation with $f'(0) = 1$ instead of $f''(0) = -1$ and is investigated in [2], [3] and [6]. This alternative set of boundary conditions leads to significant differences in the obtained results and modelizes some very different physical problem (see [7] and [8] for more details about the physical interpretation of the two sets of boundary conditions). On the other hand the blowing-up coordinates introduced to transform the differential equation (1.1) are the same as in [6], and the dynamical system obtained is very close to the one of [6]. For this reason we will refer to this paper for all that concerns the dynamical system.

The asymptotic behavior of the unbounded solutions for both prescribed surface temperature and prescribed heat flux is studied in [5].

2 Preliminary results

First, if f verifies (1.1) let us notice that

$$\left(f'' e^{(m+2)F}\right)' = (2m+1) f'^2 e^{(m+2)F} \tag{2.1}$$

with F any anti-derivative of f. As f' and f'' cannot vanish at the same point without being identically equal to zero, we deduce the

Lemma 2.1 *Let f be a non constant solution of (1.1) on some interval I. For all $t_0 \in I$ we have*

- *If $m \leq -\frac{1}{2}$, $f''(t_0) \leq 0 \Rightarrow f''(t) < 0$ for $t > t_0$.*

- *If $m > -\frac{1}{2}$, $f''(t_0) \geq 0 \Rightarrow f''(t) > 0$ for $t > t_0$.*

Proof. It follows immediately from (2.1). ∎

Let us also remark that if f is a solution of (3.1) on $[0, T)$, then for $m \leq -\frac{1}{2}$ f would be concave and for $m > -\frac{1}{2}$ it would be either concave or concave-convex.

Proposition 2.1 *For $m \leq -\frac{1}{2}$ there is only solutions to (1.1)-(1.4) if $f'(0) > 0$. Moreover, if f is a solution of (1.1)-(1.4) then*

- *f is strictly concave, increasing and $f(t) \geq -\gamma$ for all t in $[0, \infty)$.*

- *If $m \in (-2, -\frac{1}{2}]$ and $\gamma > 0$ then f becomes positive at infinity. Moreover there exists $t_0 \geq \frac{\gamma}{f'(0)}$ such that for all $t > t_0$, $f(t) > 0$.*

Proof. Since $f''(0) = -1$ and in view of lemma 2.1, $f''(t)$ would be negative for all t which shows us that f' would be decreasing and f concave. As we want to have $f'(\infty) = 0$, we must have $f'(t) > 0$ for all t.

For $m \in (-2, -\frac{1}{2}]$, using the fact that $f''(t)$ is negative for all t, we see from (1.1) that if $f(t) \leq 0$ for all t we also have $f'''(t) < 0$ for all t. This implies that f' is concave and as f' is positive we cannot have $f'(\infty) = 0$.

Finally as f is concave its graph is under its tangent in particular under that at 0 which equation is $y = f'(0)t - \gamma$. Thus f becomes positive after the point of intersection of its tangent at 0 and the t-axis, it means after $t_0 = \frac{\gamma}{f'(0)}$. ∎

Proposition 2.2 *Let f be a solution of (1.1)-(1.4). For $m > -\frac{1}{2}$ we have*

- *Either f is strictly concave and increasing and we must have $f'(0) > 0$.*

- *Or f is concave-convex and*

  - if $f'(0) \leq 0$ the solution only exists for $\gamma < 0$ and is positive and decreasing.

  - if $f'(0) \geq 0$ the solution is increasing-decreasing and positive for $t \geq t_0$ with t_0 such that $f''(t_0) = 0$.

Proof.

- As $f''(0) = -1$ if f'' does not vanish it would remain negative and f would be strictly concave. As above considering that f' would be decreasing, to have $f'(\infty) = 0$ we must have $f' > 0$.

- Suppose there exists t_0 such that $f''(t_0) = 0$ and $f'' < 0$ on $[0, t_0)$. Using lemma 2.1 we then have $f'' > 0$ on (t_0, ∞) which shows that f would be concave-convex. We also have that f' would be decreasing on $[0, t_0)$ and increasing on $[t_0, \infty)$ which implies that f' admits a negative minimum at t_0 because if not we cannot have $f'(\infty) = 0$. Thus we have the two following cases: if $f'(0) \leq 0$ then $f'(t) < 0$ on $[0, \infty)$ and f would be decreasing and if $f'(0) \geq 0$ then there exists $t_1 < t_0$ such that $f'(t_1) = 0$ and f would be increasing on $[0, t_1)$ and decreasing on $[t_1, \infty)$ which implies that f admits a maximum at t_1. If now $f(t_2) = 0$ for some $t_2 \geq t_0$, then $f(t) \leq 0$ for all $t \geq t_2$ and since $f''(t) > 0$ for $t > t_2$ we deduce from (1.1) that $f''' \geq 0$ and that f' is convex on $[t_2, \infty)$. But $f'(t_2) < 0$ and we cannot have $f'(\infty) = 0$, so $f(t) > 0$ for all $t \geq t_0$. As a consequence we cannot have a concave-convex solution with $f'(0) \leq 0$ and $\gamma > 0$.

∎

Proposition 2.3 *For $m \geq -\frac{1}{2}$ the solutions of (1.1)-(1.4) are bounded.*

Proof. For the concave-convex solutions the result is immediate. Suppose f is concave and unbounded, i.e. $f(t) \to \infty$ as $t \to \infty$. Then we have

$$f''' + (m + 2) f f'' = (2m + 1) f'^2 \geq 0$$
$$\Rightarrow \quad f''' \geq -(m + 2) f f''$$

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and using the fact that $f'' \leq 0$, if we choose t_1 such that $f(t_1) \geq \frac{1}{m+2}$ we have

$$\forall t \in [t_1, \infty), \quad f'''(t) \geq -f''(t). \tag{2.2}$$

As $f''' \geq 0$ on $[t_1, \infty)$, f'' is increasing on $[t_1, \infty)$ and using the fact that $f'(\infty) = 0$ we deduce that $f''(\infty) = 0$. Integrating (2.2) between the limits $r \geq t_1$ and ∞ leads to

$$\forall r \geq t_1, \quad -f''(r) \geq f'(r).$$

Integrating once again we obtain

$$\forall t \geq t_1, \quad -f'(t) + f'(t_1) \geq f(t) - f(t_1)$$

which means that $f'(\infty) = -\infty$ whereas one should have $f'(\infty) = 0$, a contradiction. ∎

Proposition 2.4 *For all $m \in \mathbb{R}$ if f is a solution of (1.1)-(1.4) we have*

$$\lim_{t \to \infty} f''(t) = 0.$$

Proof. See [3]. ∎

Proposition 2.5 *If a solution f of (1.1) is only defined on a finite interval $[0, T)$, then $|f(t)|$, $|f'(t)|$ and $|f''(t)|$ tends toward infinity as $t \to T$.*

Proof. See [3]. ∎

2.1   Some equalities

Integrating (1.1) on $[\rho, r]$ leads to

$$f''(r) - f''(\rho) + (m+2)f(r)f'(r) - (m+2)f(\rho)f'(\rho) = 3(m+1)\int_\rho^r f'(\xi)^2 d\xi. \tag{2.3}$$

Multiplying (1.1) by t and integrating on $[\rho, r]$ leads to

$$rf''(r) - \rho f''(\rho) - f'(r) + f'(\rho) + (m+2)(rf(r)f'(r) - \rho f(\rho)f'(\rho)) \tag{2.4}$$
$$- \frac{(m+2)}{2}\left(f(r)^2 - f(\rho)^2\right) = 3(m+1)\int_\rho^r \xi f'(\xi)^2 d\xi.$$

Multiplying (1.1) by f and integrating on $[\rho, r]$ leads to

$$f(r)f''(r) - f(\rho)f''(\rho) - \frac{1}{2}(f'(r)^2 - f'(\rho)^2) + (m+2)(f^2(r)f'(r) - f^2(\rho)f'(\rho)) \tag{2.5}$$
$$= (4m+5)\int_\rho^r f(\xi)f'(\xi)^2 d\xi.$$

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2.2 The plane dynamical system

Consider a right maximal interval $I = [\tau, \tau + T)$ on which f does not vanish. For all t in I, set

$$s = \int_\tau^t f(\xi)d\xi, \quad u(s) = \frac{f'(t)}{f(t)^2}, \quad v(s) = \frac{f''(t)}{f(t)^3}, \tag{2.6}$$

to obtain the system

$$\begin{cases} \dot{u} = P(u,v) := v - 2u^2, \\ \dot{v} = Q_m(u,v) := -(m+2)v + (2m+1)u^2 - 3uv, \end{cases} \tag{2.7}$$

in which the dot denotes the differentiation with respect to s. Let us notice that if f is negative on I then s decreases as t grows.

The singular points of (2.7) are $O = (0,0)$ and $A = \left(-\frac{1}{2}, \frac{1}{2}\right)$. The isoclinic curves $P(u,v) = 0$ and $Q_m(u,v) = 0$ are given by $v = 2u^2$ and $v = \Psi_m(u)$ where

$$\Psi_m(u) = \frac{(2m+1)u^2}{3u + (m+2)}.$$

The point A is

- An unstable node for $m \leq \frac{3-2\sqrt{6}}{2}$ ($\lambda_1 \geq 0$ and $\lambda_2 \geq 0$).

- An unstable focus if $\frac{3-2\sqrt{6}}{2} < m < \frac{3}{2}$ ($Re(\lambda_1) \geq 0$ and $Re(\lambda_2) \geq 0$).

- A center if $m = \frac{3}{2}$.

- A stable focus if $\frac{3}{2} < m < \frac{3+2\sqrt{6}}{2}$ ($Re(\lambda_1) \leq 0$ and $Re(\lambda_2) \leq 0$).

- A stable node if $m \geq \frac{3+2\sqrt{6}}{2}$ ($\lambda_1 \leq 0$ and $\lambda_2 \leq 0$).

For $m \neq -2$, the singular point O is a saddle-node of multiplicity 2. It admits a center manifold $\mathcal{W}_0$ that is tangent to the subspace $L_0 = Sp\{(1,0)\}$ and a stable (resp. unstable) manifold $\mathcal{W}$ if $m > -2$ (resp $m < -2$) that is tangent to the subspace $L = Sp\{(1, -(m+2))\}$.

We will now precise the phase portrait of the vector field (2.7) near the saddle-node point O using the same arguments as in [6] (see Fig 2.2.1).

- The parabolic sector is delimited by the separatrices S_0 and S_1 which are tangent to L at O.

- The first hyperbolic sector is delimited by S_0 and the separatrix S_2 which is tangent to L_0 at O. The second hyperbolic sector is delimited by S_1 and S_2.

- The manifold $\mathcal{W}$ is the union of the separatrices S_0, S_1 and the singular point O

$$\mathcal{W} = \{S_0\} \cup \{O\} \cup \{S_1\}.$$

Near O, the manifold $\mathcal{W}$ takes place below L for $m < -2$ or $m > -1$ and above L for $-2 < m < -1$.

In the case $m = -1$ the manifold $\mathcal{W}$ is given by $\mathcal{W} = \left\{(u, -u) \in \mathbb{R}^2 ; u > -\frac{1}{2}\right\}$.

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- The manifold $\mathcal{W}_0$ is the union of the separatrix S_2, the point O and a phase curve C_3

$$\mathcal{W}_0 = \{S_2\} \cup \{O\} \cup \{C_3\}\,.$$

Near the point O, the center manifold $\mathcal{W}_0$ takes place above L_0 for $m < -2$ or $m > -\frac{1}{2}$ and below L_0 for $-2 < m < -\frac{1}{2}$.

For $m = -\frac{1}{2}$, the center manifold $\mathcal{W}_0$ coincides with the u-axis.

Remark 2.1 *We will not consider the case $m = -2$ because we will see later that there is no solution.*

If we note S_i^+ for an ω-separatrix and S_i^- for an α-separatrix the behavior of the vector field in the neighborhood of the saddle-node point O is given by the following figures

$$m < -2 \qquad\qquad -2 < m < -1$$

$$-1 < m < -\tfrac{1}{2} \qquad\qquad m > -\tfrac{1}{2}$$
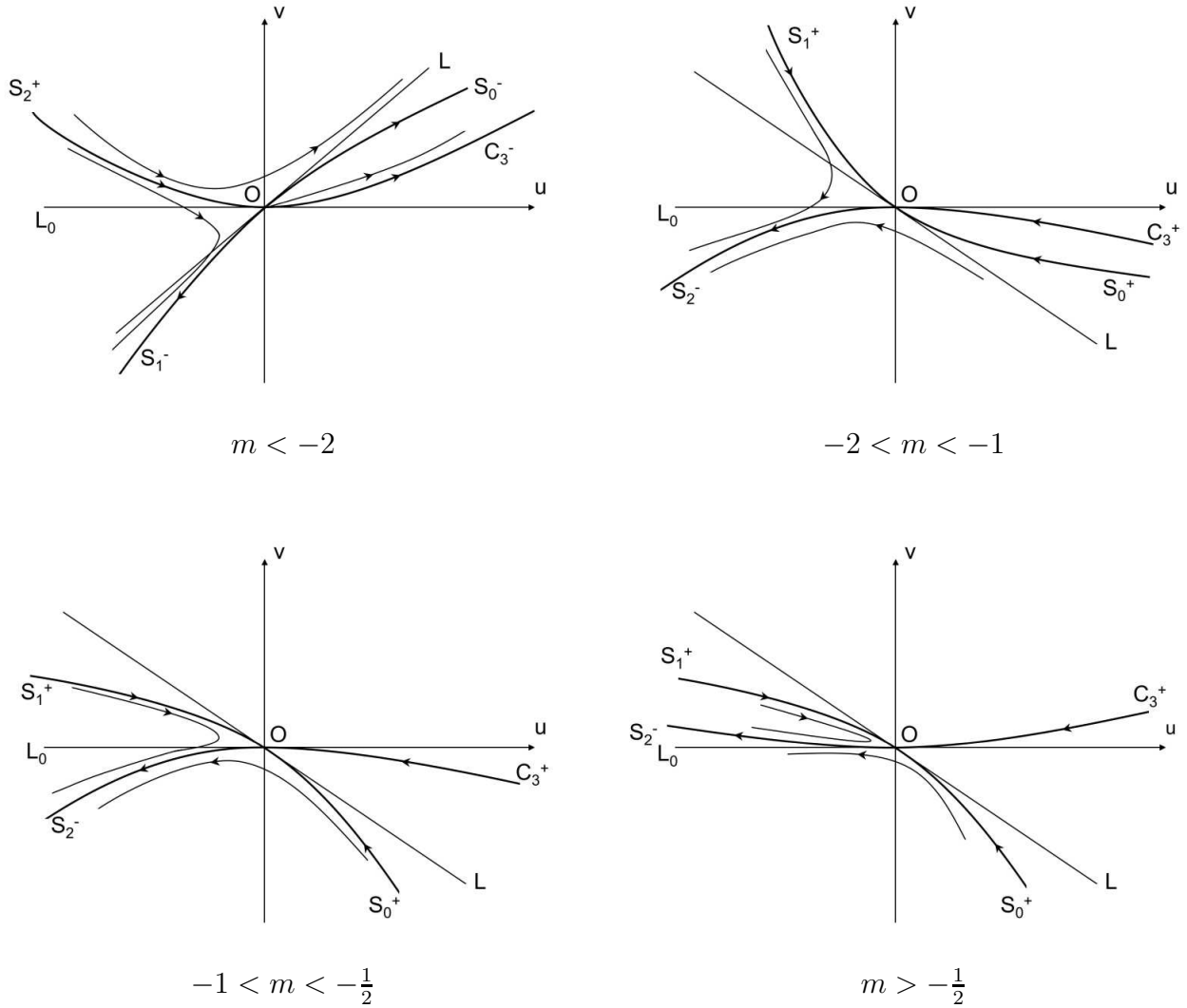
Fig 2.2.1

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We will also use the following notations: Consider a connected piece of a phase curve C of (2.7) lying in the region $P(u, v) < 0$ (resp. $P(u, v) > 0$), then C can be characterized by $v = V_m(u)$ (resp. $v = W_m(u)$) with u belonging to some interval and V_m (resp. W_m) a solution of the differential equation

$$v' = F_m(u, v) := \frac{Q_m(u, v)}{P(u, v)} = \frac{-(m+2)v + (2m+1)u^2 - 3uv}{v - 2u^2}. \tag{2.8}$$

3 Main results

To obtain results about the boundary value problem (1.1)-(1.4) we will often use the initial value problem $\mathcal{P}_{m,\gamma,\alpha}$

$$\begin{cases} f''' + (m+2)ff'' - (2m+1)f'^2 = 0, \\ f(0) = -\gamma, \\ f'(0) = \alpha, \\ f''(0) = -1 \end{cases} \tag{3.1}$$

with $\alpha \in \mathbb{R}$.

3.1 The case $m \leq -2$

Lemma 3.1 Let $m \leq -2$. If $\gamma \leq \sqrt[3]{\frac{2}{(m+2)^2}}$ the problem (1.1)-(1.4) has no solution. In particular, for $m = -2$ there is no solution at all. Moreover, if $\gamma > \sqrt[3]{\frac{2}{(m+2)^2}}$ and if f is a solution of (1.1)-(1.4), then f is negative and the phase curve $(u(s), v(s))$ defined by (2.6) with $\tau = 0$ is a negative semi-trajectory which lies for $-s$ large enough in the bounded domain

$$\mathcal{D}_+ = \left\{ (u, v) \in \mathbb{R}^2 ; \quad 0 < u < -\frac{m+2}{2} \quad \text{and} \quad 0 \leq v < -(m+2)u \right\}.$$

Proof. Suppose that f is a solution of (1.1)-(1.4). By proposition 2.1, we know that $f''(t) < 0$ and $f'(t) > 0$ for all t. If there exists t_1 such that $f(t) > 0$ for $t > t_1$, then we deduce from (1.1) that $f'''(t) \leq 0$ for $t > t_1$. This implies that f' is concave on (t_1, ∞), which does not allow to have $f'(t_1) > 0$ and $f'(\infty) = 0$. Therefore, if f is a solution of (1.1)-(1.4) we necessarily have $\gamma > 0$ and $f(t) < 0$ for all t. Next, we have

$$\forall t \geq 0, \quad \frac{f'(t)}{f(t)^2} \geq 0 \quad \text{and} \quad \frac{f''(t)}{f(t)^3} > 0. \tag{3.2}$$

As f is bounded, we can write (2.3) with $\rho = t$ and $r = \infty$ to get

$$\forall t \geq 0, \quad f''(t) + (m+2)f(t)f'(t) = -3(m+1)\int_t^\infty f'(\xi)^2 d\xi,$$

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and

$$\forall t \geq 0, \quad f''(t) + (m+2)f(t)f'(t) > 0 \tag{3.3}$$

as $m < -2$. Let λ be the limit of f at infinity, integrating again leads to

$$\forall t \geq 0, \quad f'(t) + \frac{(m+2)}{2}f(t)^2 < \frac{(m+2)}{2}\lambda^2 < 0. \tag{3.4}$$

Writing (3.3) and (3.4) for $t = 0$ we obtain

$$\gamma^2 > -\frac{2f'(0)}{m+2} \quad \text{and} \quad f'(0) > -\frac{1}{(m+2)\gamma},$$

and finally $\gamma > \sqrt[3]{\frac{2}{(m+2)^2}}$. Dividing (3.3) by $f(t)^3$ and (3.4) by $f(t)^2$ we get

$$\forall t \geq 0, \quad \frac{f'(t)}{f(t)^2} + \frac{(m+2)}{2} < 0 \quad \text{and} \quad \frac{f''(t)}{f(t)^3} + (m+2)\frac{f'(t)}{f(t)^2} < 0. \tag{3.5}$$

Using the first of the two precedent inequalities we found that

$$\forall t \geq 0, \quad f(t) \leq \frac{1}{\frac{m+2}{2}t - \frac{1}{\gamma}}$$

which implies that $\int_0^\infty f(\xi)d\xi = -\infty$. Hence the trajectory $s \mapsto (u(s), v(s))$ is defined on the whole interval $(-\infty, 0]$ and with (3.2) and (3.5) the proof is complete. ∎

In the following we will sometimes need the system (2.7) to obtain results about the problem (1.1)-(1.4) when direct approach fails. To this end we will give the behavior of the separatrices without proof because it is the same as in [6].

Theorem 3.1 *Let $m < -2$. There exists γ_* such that the problem (1.1)-(1.4) has infinitely many solutions if $\gamma > \gamma_*$, one and only one solution if $\gamma = \gamma_*$ and no solution if $\gamma < \gamma_*$.*

Proof. Taking into account proposition 2.1 and lemma 3.1, consider the solution of the initial value problem (3.1) with $\alpha > 0$ and $\gamma > 0$. Denote by $C_{\gamma,\alpha}$ the corresponding trajectory of the plane system (2.7) defined by (2.6) with $\tau = 0$. We have $u(0) = \frac{\alpha}{\gamma^2} > 0$ and $v(0) = \frac{1}{\gamma^3} > 0$.

Before going further, according to Fig 2.2.1 we just have to precise the behavior of the separatrice S_0^-. As s grows from $-\infty$, the α-separatrix S_0^- leaves to the right the singular point O tangentially to L and intersects first the isocline $Q_m(u, v) = 0$, then the isocline $P(u, v) = 0$, the u-axis and the v-axis (see Fig 3.1.1). Let (u_*, v_*) be the point where the separatrix S_0^- intersects the isocline $Q_m(u, v) = 0$ and set $\gamma_* = \frac{1}{\sqrt[3]{v_*}}$.

If $\gamma < \gamma_*$, the straight line $v = \frac{1}{\gamma^3}$ does not intersect the separatrix S_0^- and for all $\alpha > 0$, the Poincaré-Bendixson theorem shows that $C_{\gamma,\alpha}$ does not remain in the bounded domain $\mathcal{D}_+$. It follows from lemma 3.1 that f cannot be a solution of (1.1)-(1.4) for any $\alpha > 0$.

For $\gamma = \gamma_*$ the straight line $v = \frac{1}{\gamma_*^3}$ intersects the separatrix S_0^- at the point (u_*, v_*). As above, f is not a solution for $\alpha \neq \gamma^2 u_*$. For $\alpha = \gamma^2 u_*$, the phase curve $C_{\gamma,\alpha}$ is a negative semi-trajectory which coincides with the part of the separatrix S_0^- coming from O. Then f

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cannot vanish, because on the contrary one of the coordinates u or v should go to infinity (recall f' and f'' cannot vanish at the same point). Hence as long as f exists we have $f' > 0$ and $f'' < 0$, which implies that f exists on the whole interval $[0, \infty)$. Moreover $f'(t) \to l \geq 0$ as $t \to \infty$ and supposing $l > 0$ leads to a contradiction due to the negativity of f. Therefore f is a solution of (1.1)-(1.4).
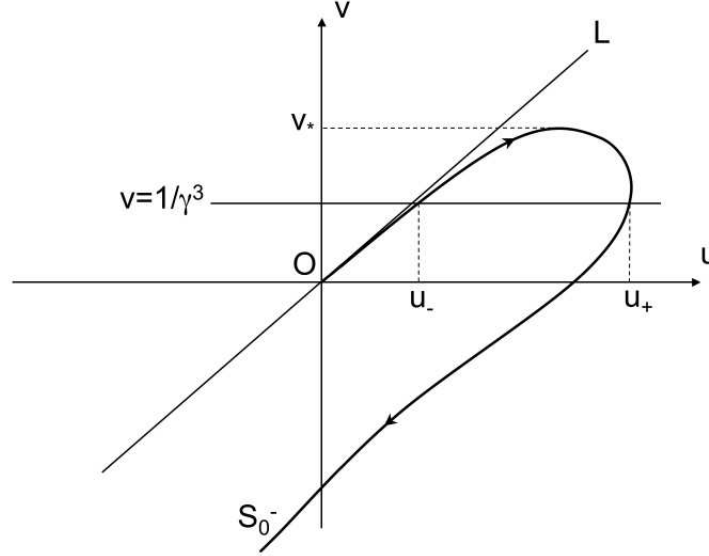
$$m < -2$$
Fig 3.1.1

For $\gamma > \gamma_*$, the straight line intersects two times the separatrix S_0^- in $\left(u_-, \frac{1}{\gamma^3}\right)$ and $\left(u_+, \frac{1}{\gamma^3}\right)$ as shown in Fig 3.1.1. Using the same arguments as above we see that if $\alpha \in [u_-\gamma^2, u_+\gamma^2]$ then f is a solution of (1.1)-(1.4) and if $\alpha \notin [u_-\gamma^2, u_+\gamma^2]$ then f is not. ■

Remark 3.1 *Notice that for all $\gamma > 0$ we have $0 < u_+ \leq \frac{1}{\sqrt{2\gamma^3}}$ (i.e. $0 < \alpha \leq \sqrt{\frac{\gamma}{2}}$) and that $u_- \to 0$ as $\gamma \to \infty$.*

Proposition 3.1 *Let $m < -2$ and f be a solution of (1.1)-(1.4), then for $\gamma = \gamma_*$ we have that $f(t) \to \lambda < 0$ as $t \to \infty$ and for every $\gamma > \gamma_*$ there are two solutions f such that $f(t) \to \lambda < 0$ as $t \to \infty$ and all the other solutions verify $f(t) \to 0$ as $t \to \infty$*

Proof. The proof is the same as in [6]. ■

3.2   The case $-2 < m < -1$

Proposition 3.2 *For $-2 < m < -1$ and $\gamma \geq 0$ the problem (1.1)-(1.4) has no solution. Moreover, to have solutions with $\gamma < 0$ we must have $f'(0) \geq -\frac{1}{(m+2)\gamma}$.*

Proof. From proposition 2.1 if f is a solution of (1.1)-(1.4) we know that $f'(0) > 0$ and that for t large enough $f(t) > 0$ and $f'(t) > 0$. Thus for $-2 < m < -1$ and $\gamma \geq 0$, we get from (2.3) with $\rho = 0$ and $r = t$

$$f''(t) = 3(m+1)\int_0^t f'^2(\xi)d\xi - (m+2)f(t)f'(t) - (m+2)\gamma f'(0) - 1 \leq -1. \qquad (3.6)$$

and a contradiction with proposition 2.4.

Let $\gamma < 0$, then for all $t > 0$ we have $f'(t) > 0$ and $f(t) > 0$. Using equality (3.6) we obtain

$$-f''(t) - (m+2)f'(0)\gamma - 1 \geq 0$$

and as t goes to infinity, using proposition 2.4 this inequality gives the second part of the result. ∎

Theorem 3.2 *For $-2 < m < -1$ there exists $\gamma_* < 0$ such that the problem (1.1)-(1.4) has no solutions for $\gamma > \gamma_*$, one and only one solution which is bounded for $\gamma = \gamma_*$ and two bounded solutions and infinitely many unbounded solutions for $\gamma < \gamma_*$.*

Proof. From proposition 3.2 we know that if $\gamma \geq 0$ there is no solution, so we must consider a solution f of the initial value problem (3.1) with $\gamma < 0$ and $\alpha > 0$. Let $C_{\gamma,\alpha}$ be the phase curve corresponding to u, v defined by (2.6) with $\tau = 0$. Looking at Fig 2.2.1, we see that the separatrix S_0^+ crosses first the u-axis, then the isocline $Q_m(u, v) = 0$ before going to O. Let us call (u_*, v_*) the point where the separatrix S_0^+ crosses the isocline $Q_m(u, v) = 0$ and set $\gamma_* = \frac{1}{\sqrt[3]{v_*}}$ (see Fig 3.2.1).
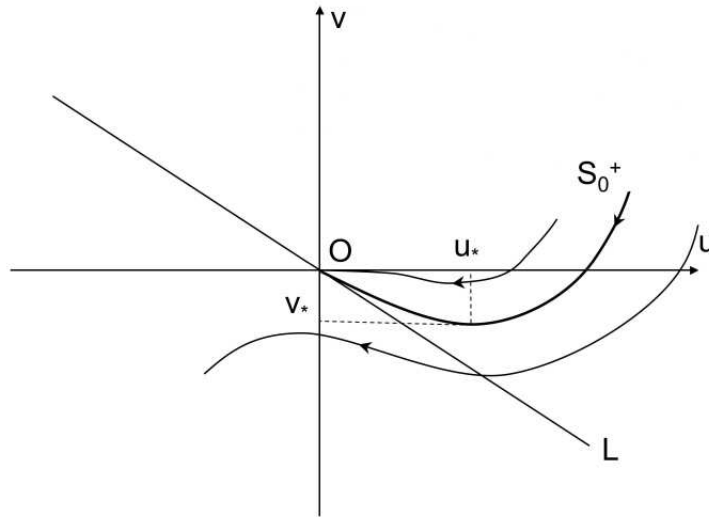
$-2 < m < -1$

Fig 3.2.1

We see that the horizontal line $v = \frac{1}{\gamma^3}$ does not intersect the separatrix S_0^+ if $\gamma_* < \gamma < 0$, is tangent to it if $\gamma = \gamma_*$, and intersects it through two points $(u_-, \frac{1}{\gamma^3})$ and $(u_+, \frac{1}{\gamma^3})$ if $\gamma < \gamma_*$. We immediately get that the problem (1.1)-(1.4) has no solutions for $\gamma > \gamma_*$. Indeed, in this case the phase curve $C_{\gamma,\alpha}$ crosses the v-axis meaning that f' vanishes and f cannot be a solution.

Let us show that if $\alpha = \gamma_*^2 u_*$ then f is a bounded solution of (1.1)-(1.4). As $C_{\gamma,\alpha}$ tends to the point O as $s \to \infty$ tangentially with the line L, we have that for t large enough $f'(t) > 0$ and $f''(t) < 0$ which implies that f is defined on the whole interval $[0, \infty)$. Furthermore

$$\frac{f'(t)}{f(t)^2} \to 0 \quad \text{and} \quad \frac{f''(t)}{f(t)f'(t)} \to -(m+2) \quad \text{as} \quad t \to \infty. \tag{3.7}$$

Hence $f'(t) \to l \geqslant 0$ as $t \to \infty$ and if we suppose $l > 0$, from (3.7) we have

$$f''(t) \sim -(m+2)l^2 t \quad \text{as} \quad t \to \infty,$$

and a contradiction with the fact that $f'(t) \to l > 0$ as $t \to \infty$. So $l = 0$ and f is a solution to (1.1)-(1.4). Suppose now that f is unbounded, i.e. $f(t) \to \infty$ as $t \to \infty$. Due to (3.7), there exists $t_0 > 0$ such that

$$\forall t \geqslant t_0, \quad f''(t) \leqslant -\frac{m+2}{2}f(t)f'(t).$$

Integrating an dividing by $f(t)^2$ leads to

$$\forall t \geqslant t_0, \quad \frac{f'(t)}{f(t)^2} - \frac{f'(t_0)}{f(t)^2} \leqslant -\frac{m+2}{4}\left(1 - \frac{f(t_0)^2}{f(t)^2}\right).$$

And using (3.7) leads to a contradiction as $t \to \infty$.

Let us now look at what happens for $u_-\gamma^2 < \alpha < u_+\gamma^2$. Because of the behavior of the vector field in the area $\{u > 0\} \cap \{v < 0\}$, we know that the phase curve $C_{\gamma,\alpha}$ has to go to the singular point O as $s \to \infty$ tangentially with the u-axis and below it. Thus, for large t we have $f'(t) > 0$, $f''(t) < 0$ and again f is defined on $[0, \infty)$. Moreover

$$\frac{f'(t)}{f(t)^2} \to 0 \quad \text{and} \quad \frac{f''(t)}{f(t)f'(t)} \to 0 \quad \text{as} \quad t \to \infty. \tag{3.8}$$

Hence $f'(t) \to l \geqslant 0$ as $t \to \infty$ and supposing $l > 0$, we get from the following equality

$$f''(t) + (m+2)f(t)f'(t) = -1 - (m+2)\gamma\alpha + 3(m+1)\int_0^t f'(\xi)^2 d\xi,$$

that

$$f''(t) \sim (2m+1)l^2 t \quad \text{as} \quad t \to \infty,$$

which contradicts the fact that $f'(t) \to l > 0$ as $t \to \infty$ and f is a solution of (1.1)-(1.4).

Let us now prove that these solutions are unbounded. If f were bounded, i.e. $f(t) \to \lambda$ as $t \to \infty$, we can write (2.5) with $\rho = t$ and $r = \infty$ in order to have

$$-f''(t)f(t) + \frac{1}{2}f'(t)^2 - (m+2)f'(t)f(t)^2 = (4m+5)\int_t^\infty f(\xi)f'(\xi)^2 d\xi.$$

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Dividing this equality by $f'(t)f(t)^2$ and using (3.8) leads to

$$\frac{4m+5}{f'(t)f(t)^2} \int_t^\infty f(\xi)f'(\xi)^2 d\xi \sim -(m+2) \quad \text{as} \quad t \to \infty,$$

and a contradiction if $m \in \left[-\frac{5}{4}, -1\right)$. If $m \in \left(-2, -\frac{5}{4}\right)$ we get

$$\int_t^\infty f(\xi)f'(\xi)^2 d\xi \sim -\frac{m+2}{4m+5}f'(t)f(t)^2 \quad \text{as} \quad t \to \infty,$$

and using the fact that $f(t) \to \lambda$ as $t \to \infty$ we obtain

$$\int_t^\infty f'(\xi)^2 d\xi \sim -\frac{m+2}{4m+5}\lambda f'(t) \quad \text{as} \quad t \to \infty.$$

We also have from (2.3)

$$\int_t^\infty f'(\xi)^2 d\xi = -\frac{1}{3(m+1)} \left(f''(t) + (m+2)f(t)f'(t) \right).$$

Combining these two equalities, we obtain

$$\frac{f''(t)}{f(t)f'(t)} \sim -\frac{(m+2)^2}{4m+5} \neq 0,$$

and a contradiction with (3.8). We conclude that f is an unbounded solution of (1.1)-(1.4). ∎

Remark 3.2 *For $-2 < m < -1$ the critical value γ_* depends on m, moreover γ_* increases from $-\infty$ to 0.*

Remark 3.3 *If f is an unbounded solution of (1.1)-(1.4) we can show that there exists a positive constant c such that*
$$f(t) \sim ct^{\frac{m+2}{1-m}} \quad \text{as} \quad t \to \infty.$$
For more details see [5].

3.3   The case $m = -1$

For $m = -1$, equation (1.1) reduces to

$$\begin{aligned} f''' + ff'' + f'^2 &= 0 \\ \Leftrightarrow \quad f''' + (ff')' &= 0, \end{aligned}$$

and integrating on $[0, t]$ leads to

$$f''(t) + f(t)f'(t) = -1 - \gamma f'(0). \tag{3.9}$$

Integrating (3.9) and taking into account the boundary conditions (1.2)-(1.4) leads to the Riccati equation

$$f'(t) + \frac{1}{2}f(t)^2 = ct + d \tag{3.10}$$

with $c = -1 - \gamma f'(0)$ and $d = f'(0) + \frac{\gamma^2}{2}$.

Proposition 3.3 *For $m = -1$, solutions of (1.1)-(1.4) only exists if $\gamma < 0$. Moreover, if it is the case we have $f'(0) \geq -\frac{1}{\gamma}$.*

Proof. Suppose that f is a solution of (1.1)-(1.4). As $m < -\frac{1}{2}$ using proposition 2.1 shows that $f'(t) > 0$ for all t. Thus

$$\forall t \geq 0, \quad f'(t) + \frac{1}{2}f(t)^2 = ct + d \geq 0$$

and $c \geq 0 \Leftrightarrow -1 \geq \gamma f'(0)$, and we deduce that $\gamma < 0$ and $f'(0) \geq -\frac{1}{\gamma}$. ∎

Theorem 3.3 *Let $m = -1$, for every $\gamma < 0$ the problem (1.1)-(1.4) admits an unique bounded solution with $f'(0) = -\frac{1}{\gamma}$ and an infinite number of unbounded solutions with $f'(0) > -\frac{1}{\gamma}$.*

Proof. Let f be the solution of (3.1) with $\alpha \geq -\frac{1}{\gamma}$. From proposition 2.1 we have that $f'' < 0$ and using equation (3.9) we deduce that f' cannot vanish. This implies that f is defined on the whole interval $[0, \infty)$ and that $f'(t)$ has a limit $l \geq 0$ as $t \to \infty$. If we suppose $l > 0$ we have that $f(t)f'(t) \to \infty$ as $t \to \infty$ and, using (3.9), that $f''(t) \to -\infty$. Then f' must become negative and this is a contradiction. Therefore $l = 0$ and f is a solution of (1.1)-(1.4).

Suppose now that f is bounded and writing (2.3) with $\rho = 0$ and $r = \infty$ we obtain that $f'(0) = -\frac{1}{\gamma}$ and the uniqueness. ∎

Remark 3.4 *Let $f'(0) = -\frac{1}{\gamma}$, then (3.10) can be integrated and we get that the unique bounded solution of (1.1)-(1.4) is given by*

$$f(t) = \frac{2\sqrt{2d}}{\dfrac{\gamma - \sqrt{2d}}{\gamma + \sqrt{2d}}e^{\sqrt{2d}t} - 1} + \sqrt{2d},$$

with $d = -\frac{1}{\gamma} + \frac{1}{2}\gamma^2$ and $\gamma < 0$.

Remark 3.5 *Let f be an unbounded solution of (1.1)-(1.4). Using (2.3) with $m = -1$, $\rho = 0$ and $r = t$ we obtain*

$$f(t)f'(t) \to -(1 + \gamma f'(0)) \quad as \quad t \to \infty$$

from which we deduce that

$$f(t) \sim \sqrt{-2(1 + \gamma f'(0))}\sqrt{t} \quad as \quad t \to \infty.$$

3.4   The case $-1 < m \leq -\frac{1}{2}$

Let us introduce the following boundary value problem studied in [2]

$$\begin{cases} \hat{g}''' + \frac{n+1}{2}\hat{g}\hat{g}'' - n\hat{g}'^2 = 0, \\ \hat{g}(0) = 0, \\ \hat{g}'(0) = 1, \\ \hat{g}'(\infty) = 0. \end{cases} \tag{3.11}$$

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Lemma 3.2 *For $n \in \left(-\frac{1}{3}, \infty\right)$, the problem (3.11) admits as solution $\hat{g}$ which is increasing, strictly concave and verifies*

$$\forall t \geq 0, \quad 0 \leq \hat{g}(t) \leq \frac{2}{\sqrt{n+1}}.$$

Proof. See [2]. ∎

Lemma 3.3 *For every $m \in (-1, \infty)$, there exists a function g strictly concave and increasing that is a solution of the problem (1.1)-(1.4) with $\gamma = 0$. Moreover we have that*

$$\forall t \geq 0, \quad 0 \leq g(t) \leq \sqrt{\frac{2g'(0)}{m+2}}.$$

Proof. Let $m \in (-1, \infty)$ and let $\hat{g}$ a solution of the problem (3.11) with $n = \frac{2m+1}{3}$, then the function g defined by

$$g(t) = a.\hat{g}(bt)$$

with

$$a = \sqrt{\frac{g'(0)}{3}} \quad \text{and} \quad b = \sqrt{3g'(0)}$$

is a solution of (1.1)-(1.4). ∎

Lemma 3.4 *For every $m \in (-1, \infty)$, the solutions g of the problem (1.1)-(1.4) with $\gamma = 0$ are such that*

$$g'(0) \geq \frac{1}{\sqrt[3]{6(m+1)}}.$$

Proof. Let $m > -1$, using equality (2.3) leads to

$$g''(t) + (m+2)g(t)g'(t) + 1 = 3(m+1)\int_0^t g'(s)^2 ds,$$

and as $0 < g'(t) \leq \alpha$ with $\alpha = g'(0)$ and $g \geq 0$ we have

$$g''(t) + 1 \leq 3(m+1)\alpha^2 t.$$

Integrating this inequality we obtain

$$\forall t > 0, \quad \frac{3(m+1)}{2}\alpha^2 t^2 - t + \alpha \geq 0$$

and $\alpha \geq \frac{1}{\sqrt[3]{6(m+1)}}$. ∎

Theorem 3.4 *Let $\gamma \in \mathbb{R}$. If $-1 < m \leq -\frac{1}{2}$, the problem (1.1)-(1.4) admits a bounded solution f. This solution is positive at infinity, increasing, strictly concave and satisfies*

$$\forall t \geq 0, \quad -\gamma \leq f(t) \leq \sqrt{\gamma^2 + 2\frac{f'(0)}{m+2}}.$$

Moreover if $\gamma \leq 0$ such a solution is unique.

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Proof of existence. Let g be the solution of the problem (1.1)-(1.4) with $\gamma = 0$ constructed in lemma 3.3.

- Case 1: $\gamma < 0$. Since for all $k > 0$ and all t_0 the function

$$f : t \to kg(kt + t_0) \tag{3.12}$$

verifies (1.1) we want to choose k and t_0 in order to have a solution of (1.1)-(1.4) with $\gamma < 0$. First let us define the function h by

$$h : t \to \frac{g(t)^3}{g''(t)}. \tag{3.13}$$

This function is well defined on $[0, \infty)$ and verify $h(0) = 0$ and $h(t) \to -\infty$ as $t \to \infty$ because of proposition 2.4 and lemma 3.3. Thus, there exists t_0 such that $h(t_0) = \gamma^3$ and choosing

$$k = -\frac{\gamma}{g(t_0)},$$

wee see that for $\gamma < 0$ the function f defined by (3.12) with these k and t_0 is a solution of (1.1)-(1.4).

- Case 2: $\gamma > 0$. Let us consider again the function h defined by (3.13). To use the previous method we now have to look at $g(t)$ for the negative values of t. Let $(-T, \infty)$ be the maximal interval of existence of g. It is easy to see that if g'' does not vanish then $T = \infty$ because if $T < \infty$, in view of proposition 2.5, we have that $g(t) \to -\infty$, $g'(t) \to \infty$ and $g''(t) \to -\infty$ as $t \to -T$. Then as $m \leq -\frac{1}{2}$ equation (1.1) give $g'''(t) \to -\infty$, a contradiction.

If g'' vanishes, let $t_1 < 0$ be such that $g''(t_1) = 0$ and $g'' < 0$ on $(t_1, 0)$. Then h is defined on $(t_1, 0]$ and $h(t) \to \infty$ as $t \to t_1$.

Suppose now that $g'' < 0$. If h is bounded on $(-\infty, 0)$, there exists $c > 0$ such that

$$\forall t < 0, \quad \frac{g''(t)}{g(t)^3} > c.$$

Multiplying by $g(t)^3 g'(t)$ and taking into account that $g < 0$ leads to

$$g''(t)g'(t) < cg(t)^3 g'(t).$$

and integrating gives

$$\forall r < t < 0, \quad g'(t)^2 - g'(r)^2 < \frac{c}{2} \left(g(t)^4 - g(r)^4 \right)$$

and finally

$$\forall r < t < 0, \quad \frac{g'(t)^2}{g(r)^4} - \frac{g'(r)^2}{g(r)^4} < \frac{c}{2} \left(\frac{g(t)^4}{g(r)^4} - 1 \right).$$

Since $g(r) \to -\infty$ as $r \to -\infty$, there exists $r_0 < 0$ such that

$$\forall r < r_0, \quad \frac{g'(r)^2}{g(r)^4} > \frac{c}{4} \Leftrightarrow \frac{g'(r)}{g(r)^2} > \frac{\sqrt{c}}{2}.$$

Integrating the last expression for $r < t < r_0$ we get

$$\forall r < t < r_0, \quad -\frac{1}{g(t)} + \frac{1}{g(r)} > \frac{\sqrt{c}}{2}(t - r),$$

and passing to the limit as $r \to -\infty$ leads to a contradiction. Thus, h is unbounded on $(-\infty, 0)$.

Therefore h is always unbounded and there exists $t_0 < 0$ such that $h(t_0) = \gamma^3$. If we choose $k = -\frac{\gamma}{g(t_0)}$ the function f given by (3.12) is a solution of (1.1)-(1.4).

From the boundedness of g we deduce that f is bounded too. Let λ be the limit of f at infinity. Using the boundedness and concavity of f for large t leads to

$$\lim_{t \to \infty} t f'(t) f(t) = \lim_{t \to \infty} t f''(t) = 0.$$

Writing (2.4) with $\rho = 0$ and $r = \infty$ leads to

$$f'(0) - \frac{m+2}{2}\left(\lambda^2 - \gamma^2\right) = 3(m+1)\int_0^\infty \xi f'(\xi)^2 d\xi > 0. \tag{3.14}$$

And the result follows from the fact that f is increasing.

∎

Proof of uniqueness.   Let $\gamma \leq 0$. First let us remark that as $m \leq -\frac{1}{2}$ if f is a solution of (1.1)-(1.4), f is increasing and strictly concave. Thus we can define a function $v = v(y)$ such that

$$\forall t \geq 0, \quad v(f(t)) = f'(t).$$

If f is bounded, there exists λ such that $f(t) \to \lambda$ as $t \to \infty$. Then v is defined on $[-\gamma, \lambda)$, is positive and we have

$$f''(t) = v(f(t))v'(f(t)),$$
$$f'''(t) = v(f(t))v'(f(t))^2 + v(f(t))^2 v''(f(t)),$$

and (1.1) leads to

$$\forall y \in [-\gamma, \lambda), \quad v'' = -\frac{1}{v}\left(v' + (m+2)y\right)v' + (2m+1). \tag{3.15}$$

We also have

$$v(-\gamma) = v(f(0)) = f'(0) = \alpha > 0, \quad v(\lambda) := \lim_{y \to \lambda} v(y) = \lim_{t \to \infty} f'(t) = 0 \quad \text{and} \quad v'(-\gamma) = -\frac{1}{\alpha}.$$

Suppose now that there are two bounded solutions f_1 and f_2 of (1.1)-(1.4) and let λ_i be the limit of f_i at infinity for $i = 1, 2$. They give v_1, v_2 solutions of equation (3.15) defined respectively on $[-\gamma, \lambda_1)$ and $[-\gamma, \lambda_2)$ such that

$$v_1(-\gamma) = \alpha_1, \ v_2(-\gamma) = \alpha_2, \quad \text{and} \quad v_1(\lambda_1) = v_2(\lambda_2) = 0.$$

Let us suppose that $\alpha_1 < \alpha_2$ and show that $\lambda_1 \leq \lambda_2$. If, on the contrary, $\lambda_1 > \lambda_2$ the function $w = v_1 - v_2$ verifies $w(-\gamma) < 0$, $w(\lambda_2) = v_1(\lambda_2) > 0$ and $w'(-\gamma) = \frac{\alpha_1 - \alpha_2}{\alpha_1 \alpha_2} < 0$. Then w admits

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a negative minimum at some point $x \in (-\gamma, \lambda_2)$. So we have $v_1(x) < v_2(x)$, $v_1'(x) = v_2'(x)$ and $v_1''(x) \geq v_2''(x)$. We also have

$$v_1''(x) - v_2''(x) = \left(\frac{1}{v_2(x)} - \frac{1}{v_1(x)} \right) (v_1'(x) + (m+2)x) \, v_1'(x), \tag{3.16}$$

and

$$(v_1'(x) + (m+2)x) \, v_1'(x) = (f_1''(t) + (m+2)f_1(t)f_1'(t)) \frac{f_1''(t)}{f_1'(t)^2}, \tag{3.17}$$

with t such that $x = f_1(t)$. As f_1 is bounded, writing (2.3) with $\rho = t$ and $r = \infty$ leads to

$$f_1''(t) + (m+2)f_1(t)f_1'(t) = -3(m+1) \int_t^\infty f_1'(\xi)^2 d\xi < 0. \tag{3.18}$$

Using this inequality and the fact that $f_1''(t) < 0$, (3.16) and (3.17) leads to $v_1''(x) < v_2''(x)$ and a contradiction. Therefore we have $\lambda_1 \leq \lambda_2$.

Now let us prove that $v_1 \leq v_2$ on $[-\gamma, \lambda_1)$. For that suppose there exists some $y \in (-\gamma, \lambda_1)$ such that $v_1(y) > v_2(y)$ and set $w = v_1 - v_2$. As $\alpha_1 < \alpha_2$, $w(-\gamma) < 0$ and using the fact that $w(\lambda_1) \leq 0$ we deduce that w admits a positive maximum at a point $x \in (-\gamma, \lambda_1)$. Thus $v_1(x) > v_2(x)$, $v_1'(x) = v_2'(x)$ and $v_1''(x) \leq v_2''(x)$.

Using inequality (3.18) and the fact that $f_1''(t) < 0$, (3.16)-(3.17) leads to $v_1''(x) > v_2''(x)$ and a contradiction. Therefore we have $v_1 \leq v_2$ on $[-\gamma, \lambda_1)$ and

$$\int_0^\infty f_1'(\xi)^2 d\xi = \int_{-\gamma}^{\lambda_1} v_1(y)dy < \int_{-\gamma}^{\lambda_1} v_2(y)dy \leq \int_{-\gamma}^{\lambda_2} v_2(y)dy = \int_0^\infty f_2'(\xi)^2 d\xi.$$

Since

$$-1 - (m+2)\gamma\alpha_i = -3(m+1) \int_0^\infty f_i'(\xi)^2 d\xi$$

we get

$$\gamma(\alpha_1 - \alpha_2) < 0$$

and as $\alpha_1 - \alpha_2 < 0$ this leads to $\gamma > 0$ and a contradiction. ∎

Remark 3.6 *Let* $m \in \left(-1, -\frac{1}{2}\right]$ *and* f *a bounded solution of (1.1)-(1.4). As* f *is strictly concave on* $[0, \infty)$ *we have* $rf'(r) < f(r) + \gamma$ *for* $r > 0$. *If* λ *denotes the limit of* f *at infinity we get*

$$\int_0^\infty rf'(r)^2 dr < \frac{1}{2}(\lambda + \gamma)^2.$$

Then as $f'(0) > 0$, *(3.14) becomes*

$$(4m+5)\lambda^2 + 6(m+1)\gamma\lambda + (2m+1)\gamma^2 > 0$$

and for $\gamma > 0$ *we have*

$$-\frac{2m+1}{4m+5}\gamma < \lambda \leq \sqrt{\gamma^2 + 2\frac{f'(0)}{m+2}}.$$

Theorem 3.5 *Let $-1 < m \leq -\frac{1}{2}$, then for $\gamma < 0$ the problem (1.1)-(1.4) admits many infinitely unbounded solutions.*

Proof. We follow an idea of [10]. Consider the initial value problem (3.1) with $\gamma < 0$ and let f_α be its solution on $[0, T_\alpha)$. Writing (2.3) with $\rho = 0$ and $r = t < T_\alpha$ leads to

$$f_\alpha''(t) + (m+2)f_\alpha(t)f_\alpha'(t) = -(m+2)\gamma\alpha - 1 + 3(m+1)\int_0^t f_\alpha'(\xi)^2 d\xi. \qquad (3.19)$$

For the remainder of the proof let us choose $\alpha \geq -\frac{1}{(m+2)\gamma}$. Then

$$\forall t \in [0, T_\alpha), \quad f_\alpha''(t) + (m+2)f_\alpha(t)f_\alpha'(t) > 0, \qquad (3.20)$$

and it follows that $f_\alpha'(t) > 0$ for all t in $[0, T_\alpha)$. Indeed, since $f_\alpha'(0) = \alpha > 0$, we should have $f_\alpha''(t_1) \leq 0$ for t_1 the first point where $f_\alpha'(t)$ vanishes that leads to a contradiction with (3.20). Using lemma 2.1 we have that $f_\alpha'' < 0$ on $[0, T_\alpha)$, then $f_\alpha'(t) \to l \in [0, \infty)$ as $t \to T_\alpha$. As f_α is strictly concave and increasing we deduce that $T_\alpha = \infty$ and $f_\alpha(t) > 0$ on $[0, \infty)$.

If $l \neq 0$ we have $f_\alpha(t) \sim lt$ as $t \to \infty$, and using (3.19) we obtain that $f_\alpha''(t) \sim -(4m+5)l^2 t$ as $t \to \infty$ that is a contradiction with $f_\alpha'(t) \sim l$ as $t \to \infty$.

Finally we get $l = 0$ and f_α verifies (1.1)-(1.4). Furthermore, from (3.19) and the choice of α we deduce that f_α is unbounded. ∎

Remark 3.7 *As in the case $-2 < m < -1$, we also have that if f is an unbounded solution of (1.1)-(1.4), there exists a positive constant c such that*

$$f(t) \sim ct^{\frac{m+2}{1-m}} \quad as \quad t \to \infty.$$

For more details see [5].

Remark 3.8 *For $m = -\frac{1}{2}$, equation (1.1) reduces to*

$$f''' + \frac{3}{2}ff'' = 0$$

which is the Blasius equation. This equation is investigated in [9] and [11] and its concave solutions are studied in [1] and [12]. See also [4].

3.5   The case $m > -\frac{1}{2}$

Theorem 3.6 *Let $\gamma \in \mathbb{R}$. For any $m \geq -\frac{1}{2}$ the problem (1.1)-(1.4) admits one and only one concave solution f which is positive at infinity and such that*

$$\forall t \geq 0, \quad -\gamma \leq f(t) \leq \sqrt{\gamma^2 + 2\frac{f'(0)}{m+2}}. \qquad (3.21)$$

Proof of existence.   Let g be the solution of (1.1)-(1.4) with $\gamma = 0$ constructed in lemma 3.3.

- Case 1: $\gamma < 0$. The same proof as in the theorem 3.4 works well in this case too.

- Case 2: $\gamma > 0$. As in theorem 3.4 we denote by $(-T, \infty)$ the maximal interval of existence of g and we again consider the function h defined by (3.13). Using lemma 2.1, g is strictly concave, increasing and h is defined on $(-T, \infty)$. Let us prove that h is unbounded on $(-T, \infty)$.

  If $T = \infty$ the reasoning used for theorem 3.4 still works, so let us suppose that $T < \infty$. Using proposition 2.5 we have that $g(t) \to -\infty$, $g'(t) \to \infty$ and $g(t)'' \to -\infty$ as $t \to -T$. Differentiating (1.1) leads to

  $$\left(g''' e^{(m+2)G}\right)' = 3m e^{(m+2)G} g' g'' \qquad (3.22)$$

  with G any anti-derivative of g. Then, as $g'''(0) = (2m+1)g'(0)^2$ using (3.22) we have that $g''' > 0$ on $(-T, \infty)$ and setting $\beta = \frac{2m+1}{m+2}$ leads to

  $$-gg'' + \beta g'^2 > 0.$$

  We deduce that the function $\phi = g'(-g)^{-\beta}$ is positive and increasing on $(-T, 0)$ and that ϕ is bounded as $t \to -T$. If h is bounded on $(-T, 0)$, there exists a positive constant c such that $h(t)^{-1} > c > 0$ and we have that

  $$\forall t < 0, \quad g''(t)g'(t) < g(t)^3 g'(t).$$

  Integrating leads to

  $$\forall r < t < 0, \quad -g'(r)^2 < g'(t)^2 - g'(r)^2 < \frac{c}{2}\left(g(t)^4 - g(r)^4\right)$$

  and

  $$\forall r < t < 0, \quad -\frac{g'(r)^2}{g(r)^4} < \frac{c}{2}\left(\frac{g(t)^4}{g(r)^4} - 1\right).$$

  If we let t going to zero we obtain that

  $$\frac{g'(r)}{g(r)^2} \geq \sqrt{\frac{c}{2}}$$

  and

  $$0 < \sqrt{\frac{c}{2}} \leq \phi(r)(-g(r))^{\beta-2} \to 0 \quad \text{as} \quad r \to -T$$

  because $\beta < 2$. This is a contradiction.

As in any case h is unbounded we conclude the same way as in theorem 3.4. ∎

Proof of uniqueness. Let f_1 and f_2 be two concave solutions of (1.1)-(1.4) such that $f_1'(0) > f_2'(0)$ and let $k = f_1 - f_2$. The function k verify $k(0) = 0$, $k'(0) > 0$, $k''(0) = 0$ and $k'(\infty) = 0$. Moreover, using proposition 2.2 we have $f_1'(0) > 0$, $f_2'(0) > 0$ and

$$k'''(0) = (2m+1)\left(f_1'(0) + f_2'(0)\right)\left(f_1'(0) - f_2'(0)\right) > 0.$$

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Then, the function k is convex near 0 and there exists $t_0 > 0$ such that $k'(t) > 0$ on $(0, t_0]$, $k''(t_0) = 0$, $k'''(t_0) \leq 0$ and $k(t_0) > 0$.

Using the fact that $f_1''(t_0) = f_2''(t_0)$ we obtain

$$k'''(t_0) = (2m + 1)k'(t_0)\left(f_1'(t_0) + f_2'(t_0)\right) - (m + 2)f_1''(t_0)k(t_0) > 0$$

wich leads to a contradiction with $k'''(t_0) \leq 0$. ∎

Lemma 3.5 *Let $m > -\frac{1}{2}$ and f be a concave-convex solution of (1.1)-(1.4). Let t_0 be the point such that $f''(t_0) = 0$, then the curve $s \mapsto (u(s), v(s))$ defined by (2.6) with $\tau = t_0$ is a positive semi-trajectory which lies in the bounded domain*

$$\mathcal{D}_- = \left\{ (u, v) \in \mathbb{R}^2 ; \quad -\frac{m + 2}{2} < u < 0 \quad and \quad 0 \leq v < -(m + 2)u \right\}.$$

Proof. In view of proposition 2.2 we know that f is positive, decreasing and convex on $[t_0, \infty)$, thus

$$\forall t \geq t_0, \quad \frac{f'(t)}{f(t)^2} < 0 \quad \text{and} \quad \frac{f''(t)}{f(t)^3} > 0. \tag{3.23}$$

As f is bounded, writing (2.3) with $\rho = t$ and $r = \infty$ we have

$$f''(t) + (m + 2)f(t)f'(t) = -3(m + 1) \int_t^\infty f'(\xi)^2 d\xi < 0, \tag{3.24}$$

and if we denote by λ the limit of f at infinity, integrating leads to

$$f'(t) + \frac{m + 2}{2}f(t)^2 > \frac{m + 2}{2}\lambda^2 \geq 0. \tag{3.25}$$

From (3.24) and (3.25) we obtain that

$$\frac{f'(t)}{f(t)^2} + \frac{m + 2}{2} > 0 \quad \text{and} \quad \frac{f''(t)}{f(t)^3} + (m + 2)\frac{f'(t)}{f(t)^2} < 0, \tag{3.26}$$

and

$$\forall t \geq t_0, \quad f(t) \geq \frac{1}{\frac{m+2}{2}(t - t_0) + \frac{1}{f(t_0)}}$$

which implies

$$\int_{t_0}^\infty f(\xi)d\xi = \infty.$$

Hence the trajectory $s \mapsto (u(s), v(s))$ is defined on the whole interval $[0, \infty)$ and using (3.23) and (3.26) leads to the result. ∎

Remark 3.9 *For $m = 1$, equation (1.1) reduces to*

$$f''' + 3ff'' - 3f'^2 = 0.$$

Let $f = g + \eta$ with $\eta > 0$, we have

$$g''' + 3\eta g'' = 3g'^2 - 3gg''.$$

Solving $g''' + 3\eta g'' = 0$ with $g(0) = -\gamma - \eta$, $g'(\infty) = 0$ and $g''(0) = -1$ leads to

$$g(t) = -\frac{1}{9\eta^2}\left(e^{-3\eta t} - 1\right) - \gamma - \eta$$

and if we choose η as the unique positive number such that $9\eta^3 + 9\gamma\eta^2 - 1 = 0$ we easily see that g satisfies $g'^2 - gg'' = 0$. It follows that f given by

$$f(t) = -\frac{1}{9\eta^2}\left(e^{-3\eta t} - 1\right) - \gamma$$

is a solution of (1.1)-(1.4). Moreover, since $f''(t) = -e^{-3\eta t} < 0$, this is the unique concave solution of (1.1)-(1.4).

Theorem 3.7 *Let $m \in \left(-\frac{1}{2}, 1\right]$, then for any $\gamma \in \mathbb{R}$ the problem (1.1)-(1.4) admits one and only one solution, which is concave.*

Proof. Taking into account proposition 2.2 and theorem 3.6, we just have to consider the case $m \in \left(-\frac{1}{2}, 1\right]$ and prove that in this case concave-convex solutions cannot exist.
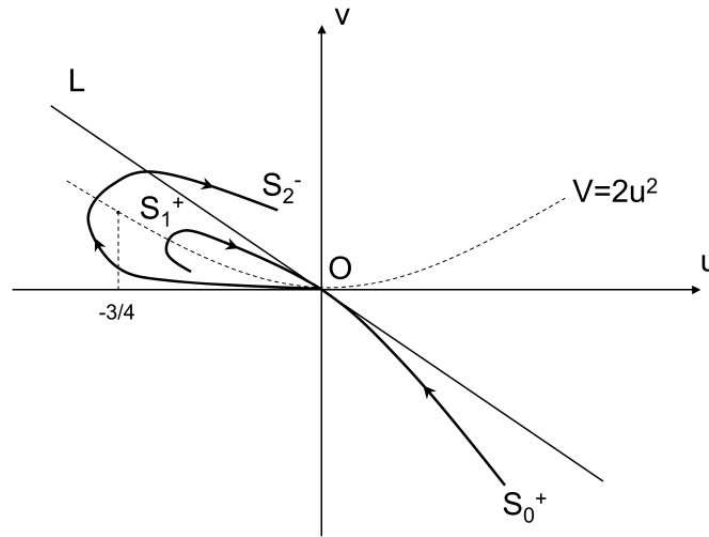
$$-\frac{1}{2} < m < 1$$
Fig 3.5.1

Suppose that f is a concave-convex solution of (1.1)-(1.4) and denote by t_0 the point where $f''(t_0) = 0$. Consider the positive semi-trajectory $s \longmapsto (u(s), v(s))$ defined in lemma 3.5, we have

$$u(0) = \frac{f'(t_0)}{f(t_0)^2} < 0 \quad \text{and} \quad v(0) = 0.$$

Refering to Fig 2.2.1 we see that the behavior of the corresponding phase curve is related to the one of the separatrices S_2^- and S_1^+.

As s increases, the separatrix S_2^- leaves the singular point O to the left tangentially with L_0, and either does not cross the isocline $P(u, v) = 0$, or crosses it through a point $(u_2, 2u_2^2)$ such that $u_2 \leq -\frac{3}{4}$ and next intersects the straight line L.
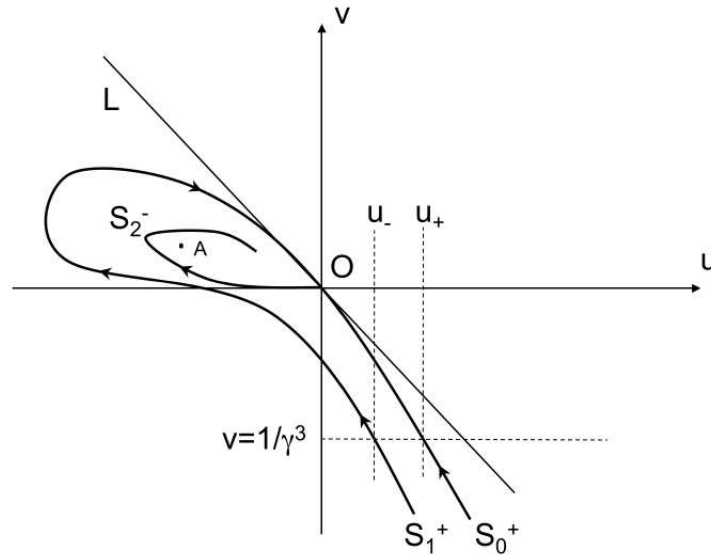
As s decreases, the separatrix S_1^+ leaves the singular point O to the left tangentially with L and crosses the isocline $P(u, v) = 0$ through a point $(u_1, 2u_1^2)$ such that $-\frac{3}{4} \leq u_1 < 0$ and next stays in the bounded region $\mathcal{D}_-$ (see Fig 3.5.1).

In view of the behavior of the separatrices we see that this semi-trajectory cannot remain in the bounded domain $\mathcal{D}_-$ and a contradiction. ∎

Theorem 3.8 *Let $m > 1$, then for any $\gamma \in \mathbb{R}$ the problem (1.1)-(1.4) has infinitely many concave-convex solutions.*

Proof.

- Case 1: Let $\gamma < 0$, and consider the initial value problem $\mathcal{P}_{m,\gamma,\alpha}$ given by (3.1) and the corresponding phase curve $C_{\gamma,\alpha}$ of the system (2.7) defined by (2.6) with $\tau = 0$. The separatrices we are concerned with, are S_0^+, S_1^+ and S_2^-.

$m > 1$

Fig 3.5.2

As s increases, the separatrix S_2^- leaves the singular point O to the left tangentially with L_0, and crosses the isocline $P(u,v) = 0$ through a point $(u_2, 2u_2^2)$ such that $-\frac{3}{4} \leq u_2 < 0$ and then stay in the bounded region $\mathcal{D}_-$.

As s decreases, the separatrix S_1^+ leaves the singular point O to the left tangentially with L and crosses the isocline $P(u,v) = 0$ through a point $(u_1, 2u_1^2)$ such that $u_1 \leq -\frac{3}{4}$. Then it intersects successively the u-axis and the v-axis and next stays in the region $\{u > 0\} \cap \{v < 0\}$ and goes to infinity with a slope that stays between $-3u - (m+2)$ and $-(m+2)$.

As s decreases, the separatrix S_0^+ leaves the singular point O to the right tangentially with L and below L. Then it stays in the region $\{u > 0\} \cap \{v < 0\}$ and goes to infinity (see Fig 3.5.2).

Looking at these separatrices we see that the straight line $v = \frac{1}{\gamma^3}$ crosses S_0^+ and S_1^+ through two points $(u_-, \frac{1}{\gamma^3})$ and $(u_+, \frac{1}{\gamma^3})$ with $u_- < u_+$.

For $\alpha \in [\gamma^2 u_-, \gamma^2 u_+)$, the trajectory $C_{\gamma,\alpha}$ intersects the u-axis for some s_0 and remains in the domain defined by the separatrix S_1^+ for $s > s_0$. It follows from the Poincaré-Bendixson Theorem that $C_{\gamma,\alpha}$ is a positive semi-trajectory whose ω-limit set is the point O if $\alpha = \gamma^2 u_-$, and either the singular point A or a limit cycle surrounding A if $\gamma^2 u_- < \alpha < \gamma^2 u_+$. Therefore f is positive as long it exists. Since $F_m(u,0) = -(m+\frac{1}{2}) < 0$ by (2.8), such a limit cycle cannot cross the u-axis and there exists $t_0 > 0$ such that $f'(t) < 0$ and $f''(t) > 0$ for $t > t_0$. Hence f is defined on $[0,\infty)$, $f'(t) \to l \leq 0$ as $t \to \infty$ and if we suppose that $l < 0$ we get a contradiction with the positivity of f. Consequently, if $\alpha \in [\gamma^2 u_-, \gamma^2 u_+)$ then f is a concave-convex solution of (1.1)-(1.4). To complete the proof in this case, let us remark that for $\alpha \notin [\gamma^2 u_-, \gamma^2 u_+]$, in view of lemma 3.5, the function f cannot be a solution of (1.1)-(1.4), and that for $\alpha = \gamma^2 u_+$ f is the concave solution.

- Case 2: Let $\gamma \geq 0$ and g be a concave-convex solution of (1.1)-(1.4) with $g(0) > 0$ and $g'(0) > 0$. Such a solution exists due to the precedent case. The function g is defined on $(-T, \infty]$ and is strictly concave on $(-T, 0]$ by lemma 2.1. Then, as $g'(0) > 0$, there exists $t_1 < 0$ such that $g(t_1) = 0$. We know that for all $k > 0$ and all t_0 the function $f(t) = kg(kt + t_0)$ verifies (1.1) and we want to choose k and t_0 to obtain a solution of (1.1)-(1.4) with $\gamma \geq 0$.

  Let us consider again the function h defined by (3.13). As g'' does not vanish on $(-T, t_1]$ h exists on $(-T, t_1]$, verifies $h(t_1) = 0$ and is unbounded. Indeed, to prove that h is unbounded, we use the same proof as in theorem 3.4 if $T = \infty$ and the same as in theorem 3.6 if $T < \infty$. Then we construct a solution of (1.1)-(1.4) with $\gamma \geq 0$ by setting $k = -\frac{\gamma}{g(t_0)}$ and the proof is complete.

∎

Remark 3.10 *Suppose given $\gamma < 0$*

- *As u_+ is the intersection of the separatrix S_0^+ that lies in the domain $\{u > 0\} \cap \{v < 0\}$ and the straight line $v = \frac{1}{\gamma^3}$ with $\gamma < 0$, we have $u_+ > 0$.*

- *If γ is such that $u_- > 0$, then all the concave-convex solutions of the problem (1.1)-(1.4) are increasing-decreasing.*

- If γ is such that $u_- < 0$, then for $\alpha \in [\gamma^2 u_-, 0]$ we get concave-convex solutions of (1.1)-(1.4) which are decreasing, and for $\alpha \in (0, \gamma^2 u_+)$ we get concave-convex solutions increasing-decreasing.

Proposition 3.4 *Let $m > 1$, then for every $\gamma \in \mathbb{R}$ there is an unique concave-convex solution that verify $f(t) \to l > 0$ as $t \to \infty$ and all the other concave-convex solutions are such that $f(t) \to 0$ as $t \to \infty$.*

Proof. Let $m > 1$ and let f be a concave-convex solution of (1.1)-(1.4). Since f is positive and decreasing at infinity, $f(t) \to \lambda \geq 0$ as $t \to \infty$. If f corresponds to the separatrix S_1^+ (i.e. $f'(0) = \gamma^2 u_-$) then we prove as in proposition 3.1 that $\lambda > 0$, and if $\gamma^2 u_- < f'(0) < \gamma^2 u_+$, there exists $c > 0$ such that $|f'(t)| > c|f(t)^2|$ for t large enough, in such a way that $\lambda = 0$. ∎

Remark 3.11 *For $1 < m < \frac{3}{2}$ the singular point A is an unstable focus, which implies that at least one cycle surrounding A has to exist. If $m > \frac{3}{2}$ then A is attractive and it seems that cycles do not exist. If it is the case, we have*

$$\frac{f'(t)}{f(t)^2} \sim -\frac{1}{2} \quad and \quad \frac{f''(t)}{f(t)^3} \sim \frac{1}{2} \quad as \quad t \to \infty,$$

which easily give

$$f(t) \sim \frac{2}{t} \quad as \quad t \to \infty.$$

4   Conclusion

- For $m < -2$ there exists $\gamma_* > \sqrt[3]{\frac{2}{(m+2)^2}}$ such that the problem (1.1)-(1.4) has no solution for $\gamma < \gamma_*$, one and only one solution for $\gamma = \gamma_*$ and infinitely many solutions for $\gamma > \gamma_*$.

  For $\gamma = \gamma_*$ we have that $f(t) \to \lambda < 0$ as $t \to \infty$ and for every $\gamma > \gamma_*$ there are two solutions f such that $f(t) \to \lambda < 0$ as $t \to \infty$ and all the other solutions verify $f(t) \to 0$ as $t \to \infty$

  Moreover, if f is a solution of (1.1)-(1.4), then f is negative, strictly concave and increasing.

- For $m = -2$ and for every $\gamma \in \mathbb{R}$, the problem (1.1)-(1.4) has no solution.

- For $-2 < m < -1$, there exists $\gamma_* < 0$ such that the problem (1.1)-(1.4) has no solution for $\gamma > \gamma_*$, one and only one solution which is bounded for $\gamma = \gamma_*$ and two bounded solutions and infinitely many unbounded solutions for $\gamma < \gamma_*$.

  Moreover, if f is a solution of (1.1)-(1.4), then f is positive, strictly concave, increasing and $f'(0) \geq -\frac{1}{(m+2)\gamma}$.

- For $m = -1$ the problem (1.1)-(1.4) only admits solutions for $\gamma < 0$. In this case there is an unique bounded solution with $f'(0) = -\frac{1}{\gamma}$ and an infinite number of unbounded solutions with $f'(0) > -\frac{1}{\gamma}$. Moreover all the solutions are positive, strictly concave and increasing.

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- For $-1 < m < -\frac{1}{2}$ the problem (1.1)-(1.4) admits at least one bounded solution for $\gamma \in \mathbb{R}$ and many infinitely unbounded solutions for $\gamma < 0$. All these solutions are increasing and strictly concave and uniqueness of the bounded solution hold for $\gamma \leq 0$.

- For $m \geq -\frac{1}{2}$ all the solutions are bounded.

- For $-\frac{1}{2} \leq m \leq 1$ and for every $\gamma \in \mathbb{R}$ the problem (1.1)-(1.4) has one and only one solution. This solution is strictly concave and increasing.

- For $m > 1$ and $\gamma \in \mathbb{R}$ the problem (1.1)-(1.4) has one and only one concave solution and infinitely many concave-convex solutions. Moreover, there is an unique concave-convex solution that verifies $f(t) \to \lambda > 0$ as $t \to \infty$ and all the other concave-convex solutions are such that $f(t) \to 0$ as $t \to \infty$.

After this study it remains to investigate the following situations

- For $-1 < m < -\frac{1}{2}$ and $\gamma > 0$ is the bounded solution unique ?

- For $-1 < m < -\frac{1}{2}$ and $\gamma \geq 0$ is there unbounded solution ?

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